UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 2ND FEBRUARY 2017

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

For reasons of space, these solutions are necessarily brief. Extended solutions, and some exercises for further investigation, can be found at:

http://www.ukmt.org.uk/

The UKMT is a registered charity

---

1. D $\frac{2}{5} + \frac{2}{50} + \frac{2}{500} = \frac{200 + 20 + 2}{500} = \frac{222}{500} = \frac{444}{1000} = 0.444.$

2. B When one third of the circle is shaded, the angle at the centre of the shaded sector is $360° \div 3 = 120°$. In diagram A, the sector angle is $90°$. In diagram C, the sector angle is $90° + 90° \div 2 = 135°$. So the correct diagram has a sector angle greater than that shown in A, but smaller than that shown in C. The only such sector angle is that in B.

3. A Consider the units digits of the squares of the integers 0, 1, 2, 3, ..., 9. These are 0, 1, 4, 9, 6, 5, 6, 9, 4, 1. Note that none of these is 7, so no square ends in a 7.

4. D All primes except 2 are odd. So the sum of a pair of primes cannot be odd, and so cannot be prime, unless one of the pair is 2. We note that $5 = 2 + 3$; $7 = 2 + 5$; $9 = 2 + 7$; $13 = 2 + 11$. However, $11 = 2 + 9$ and so it is not the sum of two primes, as 9 is not prime.

5. B Let the area of the inner circle be A. The radius of the outer circle is twice that of the inner circle, so its area is four times that of the inner circle , that is $4A$. Therefore the region between the two circles has area $3A$. As three of the six equal segments are shaded, the total shaded area is $3A \div 2$. So the required fraction is $\frac{3}{2}A \div 4A = \frac{3}{8}$.

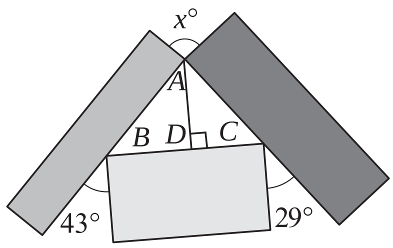
6. D The sum of the interior angles of a quadrilateral is 360°. So the smallest angle in the quadrilateral is $3 \times \frac{360°}{3+4+5+6} = 60°$ and the largest angle is $6 \times \frac{360°}{3+4+5+6} = 120°$. Hence the required difference is 60°.

7. C The smallest possible range of four positive integers is 3. Let these integers be n, $n + 1$, $n + 2$, $n + 3$. For the mean of these four integers to equal 2017, we have $\frac{n+n+1+n+2+n+3}{4} = 2017$, that is $n = 2015\frac{1}{2}$. So the smallest possible range is not 3. However, note that the integers 2015, 2016, 2018 and 2019 have mean 2017 and range 4. So the smallest possible range of the integers is 4.

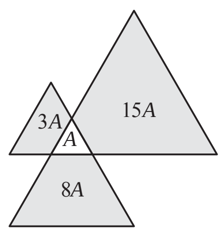
8. D Consider options B to E: $1.354\dot{2} = 1.354222\ldots$; $1.35\dot{4}\dot{2} = 1.35424242\ldots$; $1.3\dot{5}4\dot{2} = 1.3542542\ldots$; $1.\dot{3}54\dot{2} = 1.35423542\ldots$. These are all greater than 1.3542, so option A is not correct. The other four options have the same units digit and the first four digits after the decimal point are the same in all. So the largest option is that which has the largest digit five places after the decimal point and this is $1.3\dot{5}4\dot{2}$.

9. B First note that as 'ab' and 'ba' are two digit numbers, neither a nor b is equal to zero. Now 'ab' – 'ba' = $(10a + b) - (10b + a) = 9a - 9b = 9(a - b)$. So for the difference to be as large as possible, a must be as large as possible, that is 9, and b must be as small as possible, that is 1.
So the required difference is $91 - 19 = 72$.

10. A As shown in the diagram, the perpendicular from A to BC meets BC at D. Because AD is parallel to two sides of the lower rectangle, $\angle BAD = 43°$ and $\angle DAC = 29°$ (corresponding angles in both cases). The angles at point A sum to 360°, so $43 + 29 + 90 + x + 90 = 360$ and hence $x = 108$.



11. E Let the area of the equilateral triangle of side 1 be A. If similar figures have sides which are in the ratio $k : 1$, then the ratio of their areas is $k^2 : 1$. So the areas of the triangles with sides 2, 3, 4 are $4A$, $9A$, $16A$ respectively.
So, as shown in the diagram, the total shaded area is $3A + 8A + 15A = 26A$. Therefore $n = 26$.



12. D Let the ages of Alice, Bob, Clare, Dan and Eve be a, b, c, d, e respectively. So, for example, $b + c = 40$ and $d + e = 44$. Adding these gives $b + c + d + e = 84$. We are also told that $a + b + c + d + e = 105$. The difference between these equations shows that $a = 105 - 84 = 21$. Hence $b = 39 - 21 = 18$ and, similarly, $c = 40 - 18 = 22$, $d = 38 - 22 = 16$ and $e = 44 - 16 = 28$. So Dan is the youngest.

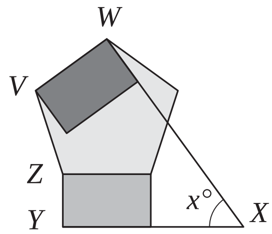
13. A Pythagoras' Theorem shows that $PR = \sqrt{3^2 + 4^2} = 5$. So the perimeter of triangle PQR is 12. Since the triangles are similar and $PR : PQ = 5 : 3$ we see that the perimeter of triangle PRS is 20. Hence the perimeter of $PQRS$ is $12 + 20 - 2 \times PR = 32 - 10 = 22$.

14. B Note that $64 = 2^6$ and $512 = 2^9$. Therefore $(2^6)^x = (2^9)^5$. So $2^{6x} = 2^{45}$. Hence $6x = 45$, that is $x = 7.5$.

15. C Let $\angle QRS = x°$. Then, as $SQ = QR$, $\angle RSQ = x°$ also. Now $\angle SPQ = 2 \times \angle RSQ = 2x°$. In triangle PQS, $PQ = SQ$, so $\angle PSQ = \angle SPQ = 2x°$. Therefore $\angle PSR = \angle PSQ + \angle RSQ = 2x° + x° = 3x°$. The sum of the interior angles of a triangle is 180°. So, considering triangle PSR, $\angle SPR + \angle PSR + \angle PRS = 180°$. Therefore $2x° + 3x° + x° = 180°$. So $6x = 180$, that is $x = 30$.

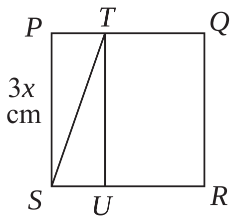
16. D Let the two positive integers be m and n. Then $mn = 2(m + n) = 6(m - n)$. So $2m + 2n = 6m - 6n$, that is $8n = 4m$. Therefore $m = 2n$. Substituting for m gives: $(2n)n = 2(2n + n)$. So $2n^2 = 6n$, that is $2n(n - 3) = 0$. Therefore $n = 0$ or 3. However, n is positive so the only solution is $n = 3$. Therefore $m = 2 \times 3 = 6$ and $m + n = 6 + 3 = 9$.

17. B Since $VWXYZ$ is a pentagon, the sum of its interior angles is 540°. Now $\angle ZVW$ is an interior angle of a regular pentagon and so is 108°. Both $\angle VWX$ and $\angle XYZ$ are 90°; and the reflex angle $\angle YZV = 90° + 108°$. Therefore $540° = 108° + 90° + 90° + 90° + 108° + x°$. Hence $x = 540 - 486 = 54$.



18. E Let the capacity of the tank be x litres. Then $30 = \frac{5x}{6} - \frac{4x}{5} = \frac{25x - 24x}{30} = \frac{x}{30}$. So $x = 30 \times 30 = 900$.

19. E Let square $PQRS$ have side $3x$ cm. Then, as $PT : TQ = 1 : 2$, $PT = x$ cm. Similarly, $US = x$ cm. In triangles PTS and UST, $PT = US$, $\angle PTS = \angle UST$ (alternate angles) and TS is common to both. So the triangles are congruent (SAS). Therefore $UT = PS = 3x$ cm and $\angle TUS = \angle SPT = 90°$. Hence $PTUS$ is a rectangle, which has perimeter 40 cm. So $40 = 2(3x + x) = 8x$. Therefore $x = 5$ and the area of $PTUS$, in cm^2, is $15 \times 5 = 75$.



20. D First note that as the length of each side of the heptagon is 4, the radius of each of the seven arcs is 2. The sum of the interior angles of a heptagon is $(7 - 2) \times 180° = 900°$. So the sum of the angles subtended by the circular arcs at the centres of the circles of radius 2 cm is $7 \times 360° - 900° = \left(7 - \frac{5}{2}\right) \times 360° = \frac{9}{2} \times 360°$. Therefore the total shaded area is equal to the total area of $\frac{9}{2}$ circles of radius 2. So the total shaded area is $\frac{9}{2} \times \pi \times 2^2 = 18\pi$.

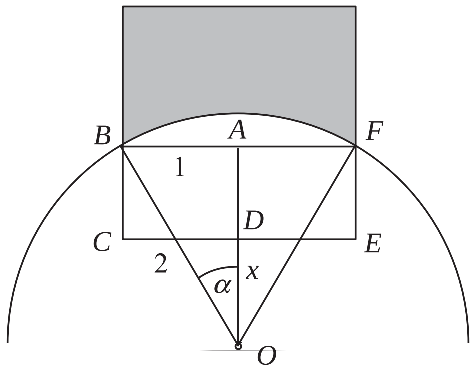
21. A Let the number of Brachycephalus frogs and common frogs in the bucket be b and c respectively. Note that each Brachycephalus frog has 6 toes and 4 fingers, while a common frog has 10 toes and 8 fingers.
Therefore, $6b + 10c = 122$ (1); $4b + 8c = 92$ (2). Subtracting (2) from (1) gives $2b + 2c = 30$, so $b + c = 15$.

22. C Consider triangles MON and MPN. Note that $MO = MP = 4$ because M is the midpoint of OP; $\angle OMN = \angle PMN = 90°$ because MN is perpendicular to OP; side NM is common to both triangles. So triangles MON and MPN are congruent (SAS). Therefore $PN = ON = 8$ because ON is a radius of the circle. By Pythagoras' Theorem in triangle OMN, $ON^2 = OM^2 + MN^2$, so $8^2 = 4^2 + MN^2$. Therefore $MN^2 = 8^2 - 4^2 = 48$. So $MN = \sqrt{48}$.
The perimeter of triangle PNM is $PN + NM + MP = 8 + \sqrt{48} + 4 = 12 + \sqrt{48}$.
Now $6.5 < \sqrt{48} < 7$, since $\sqrt{42.25} < \sqrt{48} < \sqrt{49}$. So $\sqrt{48}$ is closer in value to 7 than it is to 6. So $12 + \sqrt{48}$ is nearer to 19 than it is to 18.

23. D Let the five positions in the photograph be numbered 1, 2, 3, 4, 5. Then the boys may occupy a total of six positions: 1 and 3; 1 and 4; 1 and 5; 2 and 4; 2 and 5; 3 and 5. For each of these positions, the boys may be arranged in two ways as they can interchange places. So there are 12 ways of positioning the boys. For each of these, the girls must be placed in three positions. In each case, the first girl may choose any one of three positions, the second girl may choose either of two positions and then there is just one place remaining for the third girl. So for each arrangement of the two boys there are $3 \times 2 \times 1$ different ways of arranging the three girls. Therefore the total number of line-ups is $12 \times 6 = 72$.

24. C The product may be written $\sqrt{\frac{3}{2}} \times \sqrt{\frac{4}{3}} \times \sqrt{\frac{5}{4}} \times \sqrt{\frac{6}{5}} \times \ldots = \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{4}}{\sqrt{3}} \times \frac{\sqrt{5}}{\sqrt{4}} \times \frac{\sqrt{6}}{\sqrt{5}} \times \ldots$.
Notice that the numerator of each fraction is cancelled out by the denominator of the following fraction and the only terms which are not cancelled are the denominator of the first fraction and the numerator of the last fraction. So the nth term of the sequence is $\frac{\sqrt{n+2}}{\sqrt{2}} = \sqrt{\frac{n+2}{2}}$. As $n \geqslant 1$ the product is not equal to 1.
The product increases with n. So the next possible integer value to consider is 2 and this does occur when $n = 6$ as $\sqrt{\frac{6+2}{2}} = \sqrt{4} = 2$.
So the smallest number of terms required for the product to be an integer is 6.

25. A The diagram shows part of the diagram in the question. Point O is the centre of the circle and points A, B, C, D, E, F, are as shown. Consider triangle BOF: BF is equal in length to the side of the square so $BF = 2$. Also $OB = OF = 2$ as they are both radii of the circle. So triangle BOF is equilateral. Therefore $\angle BOF = 60°$, so the area of sector $OBF = \frac{60}{360} \times \pi \times 2^2 = \frac{2\pi}{3}$.



By Pythagoras' Theorem: $OA = \sqrt{2^2 - 1^2} = \sqrt{3}$.
So the area of triangle $OBF = \frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$. Therefore the area of the segment bounded by arc BF and line segment $BF = \frac{2\pi}{3} - \sqrt{3}$. The area of rectangle $BFEC = BF \times AD = 2 \times (\sqrt{3} - x)$.
The shaded region has area 2, so the area of the above segment + area of rectangle $BFEC$ = area of the given square minus 2 = $4 - 2 = 2$.
Hence $\frac{2\pi}{3} - \sqrt{3} + 2\sqrt{3} - 2x = 2$. So $x = \frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1$.